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# Exceptional points of non-Hermitian operators

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**Abstract**
Exceptional points associated with non-Hermitian operators, i.e. operators being non-Hermitian for real parameter values, are investigated. The specific characteristics of the eigenfunctions at the exceptional point are worked out. Within the domain of real parameters the exceptional points are the points where eigenvalues switch from real to complex values. These and other results are exemplified by a classical problem leading to exceptional points of a non-Hermitian matrix.

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## 1. Introduction

Exceptional points are branch point singularities of the spectrum and eigenfunctions, which occur generically when a matrix, or for instance a Hamilton operator, is analytically continued in a parameter on which it depends. The term 'exceptional points' has been introduced by Kato [1]. When a physical problem is formulated by $H_0 + \lambda H_1$ with $\lambda$ being a strength parameter, the spectrum and eigenfunctions, $E_n(\lambda)$ and $|\psi_n(\lambda)\rangle$, are in general analytic functions of $\lambda$. At certain points in the complex $\lambda$-plane two energy levels coalesce. Such coalescence is not to be confused with a genuine degeneracy, since the eigenspace of the two coalescing levels is not two but only one dimensional; in fact the corresponding eigenvectors also coalesce and there is no two-dimensional subspace as is the case for a genuine degeneracy.

If both operators, $H_0$ and $H_1$, are real symmetric, these singularities—the exceptional points (EP)—can occur only for complex parameter values $\lambda$. As a consequence, at an EP the full problem $H_0 + \lambda H_1$ is no longer Hermitian as such, but when dealing with matrices, it is still complex symmetric. These cases have been studied in some detail [2, 3] and here we quote the major results.

EPs are always found in the vicinity of a level repulsion. Suppose that two levels show avoided level crossing when $\lambda$ is varied along the real axis; then the analytic continuation into the complex $\lambda$-plane yields a complex conjugate pair of EPs where the two coalescing levels are analytically connected by a square root branch point [4]. The occurrence of EPs is not restricted to repulsions of bound states, a recent paper deals with the repulsion of resonant states [5]. Being singularities in the interaction strength EPs determine the convergence radius of approximation schemes in the theory of effective interactions [6]. Quantum mechanical phase transitions are characterized by a multitude or accumulation points of EPs [7, 8].

There are a number of phenomena, where the physical effect of an EP has been at least indirectly observed. Laser-induced ionization states of atoms [9] are a clear manifestation of an EP even though in [9], it has not been analysed as such. A recent theoretical paper [10] shows that, for a suitable choice of parameters associated with an EP, the only acoustic modes in an absorptive medium are circular polarized waves with one specific orientation for a given EP. Similarly in optics, experimental observations in absorptive media [11] reveal the existence of handedness since the stable mode of light propagation is either a left or a right circular polarized wave for appropriately chosen parameters. This has been interpreted in [12] in terms of singular behaviour of the dielectric tensor. A particular resonant behaviour of atom waves in crystals of light [13] has likewise been interpreted [14] by the same mathematical mechanism. Models for Stark resonances in atomic physics have been analysed in terms of EPs and their connection to diabolic points discussed [15]. While absorption is essential in all cases, some situations clearly point to a chiral behaviour of an EP. In fact, the wavefunction at the EP has been shown to have definitive chiral character [16] and this has been experimentally confirmed recently [17].

In a previous experiment [18], EPs have been investigated in a flat microwave cavity. Major findings have been the confirmation of a fourth-order branch point of the coalescing wavefunctions and—depending on the path in the complex $\lambda$-plane—level avoidance associated with width crossing or level crossing with width avoidance. These results are the consequence of the topological structure of Riemann sheets at a branch point [4]. The experiment thus showed that this topology is a physical reality.

In the following, we carry the analysis further in that we investigate the EPs of $H_0 + \lambda H_1$ when $H_0$ or $H_1$ or both are no longer Hermitian. As a consequence, even for real values of $\lambda$, the problem $H_0 + \lambda H_1$ is no longer Hermitian. This lack of Hermiticity is different in nature from that discussed above where dissipation—like, for instance, in the optical model in nuclear physics [19]—makes $H_0 + \lambda H_1$ non-Hermitian for complex $\lambda$. There is a great variety of problems in the literature where either the perturbation or the full Hamiltonian is non-Hermitian. Boson mapping [20], effective interactions [21] and the random phase approximation (RPA) in many-body theory [22] yield non-Hermitian operators. More recently a wider class of non-Hermitian Hamiltonians has been proposed to address specific symmetries [23] or transitional points in specific delocalization models [24]. These suggestions have led to a further thorough study [25] of non-Hermitian operators.

The present study is motivated by one of the simplest problems in physics, that is the classical problem of two coupled damped oscillators. It gives rise to non-Hermitian matrices in a natural way. The problem is stated in the following section. The ensuing general treatment of section 3 yields new insights and special features regarding level repulsion. It is shown that the change from a complex to a pure real spectrum of a (real) non-Hermitian matrix under variation of the (real) parameter $\lambda$ is due to the occurrence of a real EP. As expected, the coalescence at the EP of two complex eigenvalues into one real eigenvalue (which then bifurcates in two real eigenvalues) yields only one eigenfunction in contrast to the usual two for a genuine degeneracy. A typical example is the instability point of the RPA. In addition, the pattern of level repulsion is distinctly different from that of Hermitian $H_0$ and $H_1$: the levels approach each other in the form of a cusp and not in a smooth way as is the case for Hermitian $H_0$ and $H_1$. These general findings are illustrated in section 4 where the example of section two is resumed.

We stress that the present paper focuses upon EPs and not on the study of non-Hermitian operators as such. A summary and suggestion is given in the last section.

## 2. Two coupled damped oscillators

As a first illustration, we consider a simple classical case of two damped coupled oscillators in one dimension. Denoting by $p_1, p_2, q_1, q_2$ the momenta and spatial coordinates of two point particles of equal mass the equations of motion read for the driven system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \mathcal{M}\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \exp(\mathrm{i}\omega t) \tag{1}$$

with

$$\mathcal{M} = \begin{pmatrix} -2g-2k_1 & 2g & -f-\omega_1^2 & f \\ 2g & -2g-2k_2 & f & -f-\omega_2^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2}$$

where $\omega_j - \mathrm{i}k_j$, $j = 1, 2$ are essentially the damped frequencies without coupling and $f$ and $g$ are the coupling spring constant and damping of the coupling, respectively. The driving force is assumed to be oscillatory with one single frequency and acting on each particle with amplitude $c_j$. Here we are interested only in the stationary solution being the solution of the inhomogeneous equation which reads

$$\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = (\mathrm{i}\omega - \mathcal{M})^{-1}\begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix} \exp(\mathrm{i}\omega t). \tag{3}$$

Resonances occur for the real values $\omega$ of the complex solutions of the secular equation

$$\det|\mathrm{i}\omega - \mathcal{M}| = 0 \tag{4}$$

and EPs occur for the complex values $\omega$ where

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\det|\mathrm{i}\omega - \mathcal{M}| = 0 \tag{5}$$

is fulfilled simultaneously together with equation (4). We choose the parameter $f$ as the second variable needed to enforce the simultaneous solution of equations (4) and (5) and keep the other parameters of $\mathcal{M}$ fixed, but of course any other preference—like choosing $g$—would be just as good and not alter the essential results. Thus we encounter the problem of finding the EPs of the matrix problem

$$\mathcal{M}_0 + f\mathcal{M}_1 \tag{6}$$

with

$$\mathcal{M}_0 = \begin{pmatrix} -2g-2k_1 & 2g & -\omega_1^2 & 0 \\ 2g & -2g-2k_2 & 0 & -\omega_2^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{7}$$

and

$$\mathcal{M}_1 = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

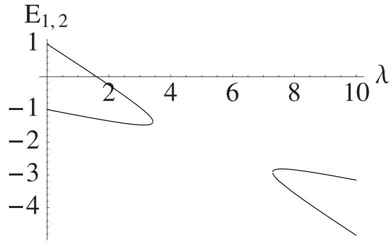


**Figure 1.** Real spectrum in arbitrary units as a function of λ. In terms of equation (9) the parameters are $\epsilon_1 = -1, \epsilon_2 = 1, \omega_1 = -0.2, \omega_2 = -0.6, \phi_1 = -2^0, \phi_2 = 45^0$. The spectrum is complex between the EPs at $\lambda_{\rm EP}^{+} = 7.3\ldots$ and $\lambda_{\rm EP}^{-} = 3.4\ldots$.

Note that $\mathcal{M}_0$ and $\mathcal{M}_1$ are not symmetric. Before we turn to explicit solutions and characteriztics of equations (4) and (5) we first address the general problem of EPs of non-Hermitian matrices.

## 3. General non-Hermitian case

Like for the Hermitian operators the behaviour around an EP can be described locally by a $2\times 2$ matrix. The reduction of an $N$-dimensional to a two-dimensional problem is given below. We always consider a situation where the unperturbed problem, denoted by $H_0$ or $h_0$ for the two-dimensional case, is assumed to be diagonal. At first we discuss the two-dimensional case and assume that the Jordan decomposition of the non-Hermitian perturbation $h_1$, i.e. $h_1 = SJS^{-1}$, yields a diagonal matrix $J$. We thus consider

$$h(\lambda) = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} + \lambda S \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} S^{-1} \tag{9}$$

with

$$S = \begin{pmatrix} \cos\phi_1 & -\sin\phi_2 \\ \sin\phi_1 & \cos\phi_2 \end{pmatrix}. \tag{10}$$

Note that for $h_1$ to be symmetric we would have $\phi_1 = \phi_2$, i.e. $S$ would be orthogonal. For convenience, we have exploited the freedom to use normalized column vectors in $S$. The two eigenvalues of $h$ are given by

$$E_{1,2}(\lambda) = \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \lambda(\omega_1 + \omega_2) \pm D) \tag{11}$$

with the discriminant

$$\begin{aligned} D = ((\epsilon_1 - \epsilon_2)^2 + \lambda^2(\omega_1 - \omega_2)^2 \\ + 2\lambda(\epsilon_1 - \epsilon_2)(\omega_1 - \omega_2)\cos(\phi_1 + \phi_2)\sec(\phi_1 - \phi_2))^{1/2}. \end{aligned} \tag{12}$$

The two levels coalesce when $D = 0$ that is for

$$\lambda_{\rm EP}^{\pm} = -\frac{\epsilon_1 - \epsilon_2}{\omega_1 - \omega_2}\big(\cos(\phi_1 + \phi_2) \pm {\rm i}\sqrt{\sin 2\phi_1 \sin 2\phi_2}\big)\sec(\phi_1 - \phi_2). \tag{13}$$

Note that even when all parameters are real the two EPs can now occur on the real axis. It happens when the signs of $\phi_1$ and $\phi_2$ are different. The implication is that the spectrum is no longer real when $\lambda$ lies between $\lambda_{\rm EP}^{-}$ and $\lambda_{\rm EP}^{+}$. In figure 1 we display a typical case of a spectrum of that nature[1]. We recall that at the EPs, where the real spectrum ends or begins, only *one* eigenfunction exists of the two by two matrix problem; its precise form is given below. Here we stress a general property of a matrix at an EP: the Jordan decomposition of $h(\lambda_{\mathrm{EP}}) = TJT^{-1}$ yields a non-diagonal matrix $J$ given by the standard form

$$J_{\mathrm{EP}} = \begin{pmatrix} E(\lambda_{\mathrm{EP}}) & 1 \\ 0 & E(\lambda_{\mathrm{EP}}) \end{pmatrix} \tag{14}$$

whereas for all points $\lambda \neq \lambda_{\mathrm{EP}}$ the matrix $J_\lambda$ is diagonal and reads

$$\begin{aligned} J_\lambda &= T^{-1}h(\lambda)T \\ &= \begin{pmatrix} E_1(\lambda) & 0 \\ 0 & E_2(\lambda) \end{pmatrix}. \end{aligned} \tag{15}$$

We recall that $T$ is orthogonal (or unitary) only if $h(\lambda)$ is Hermitian.

While it is well known that a non-Hermitian operator can have a non-real spectrum, the deviation from the Hermitian case has, for the real part of the spectrum, distinct consequences for the shape of level repulsions; this is exemplified in the following section where typical results of the two oscillators introduced in the previous section are presented.

Yet the local behaviour at the EP is basically the same as for the Hermitian case. It is clear from equations (11) and (12) that the two eigenvalues are connected at the square root branch points situated at $\lambda = \lambda_{\mathrm{EP}}^{\pm}$ just as in the Hermitian case. The difference arises in the eigenfunction at the EP. Recall that the coalescence of two eigenvalues at the EP is not to be confused with a true degeneracy in that there is only one eigenfunction at the EP. At $\lambda_{\mathrm{EP}}^{\pm}$ this single and unique eigenfunction (up to a possible common factor) turns out now to be

$$\left|\psi_{\mathrm{EP}}^{\pm}\right\rangle = \pm\mathrm{i}\sqrt{\frac{\cot\phi_1}{\cot\phi_2}}|1\rangle + |2\rangle. \tag{16}$$

We note that, in contrast to the symmetric case, the left-hand eigenfunction at the EP (or the eigenfunction of the adjoint problem) is now different and reads

$$\left\langle\psi_{\mathrm{EP}}^{\pm}\right| = \pm\mathrm{i}\sqrt{\frac{\tan\phi_1}{\tan\phi_2}}\langle 1| + \langle 2|. \tag{17}$$

As a result, the relation

$$\left\langle\psi_{\mathrm{EP}}^{+}\middle|\psi_{\mathrm{EP}}^{+}\right\rangle = \left\langle\psi_{\mathrm{EP}}^{-}\middle|\psi_{\mathrm{EP}}^{-}\right\rangle = 0 \tag{18}$$

still prevails just as in the symmetric case. The basis vectors $|j\rangle$, $j = 1, 2$ refer to the eigenstates of $h_0$.

We only mention the special case where either $\phi_1$ or $\phi_2$ assumes the value 0 or $\pi/2$: in contrast to the Hermitian case the confluence of the two EPs does not invoke a true degeneracy with two independent eigenvectors even though it gives rise—for real parameters—to a real level crossing; also equation (18) is upheld at such points.

To summarize, for real matrix elements and different signs of the angles, the spectrum is complex between the two real EPs. Regarding the wavefunction the quotient of the amplitudes of the two coalescing wavefunctions deviates from that of the Hermitian case. The genuinely complex superposition of the eigenfunction at the EP remains, however. For real angles $\phi_j$ (of equal sign), the fixed phase difference of $\pm\pi/2$ between the basis states at $\lambda_{\mathrm{EP}}^{\pm}$ occurs just as in the symmetric case, but the ratio of the modulus of the amplitudes is in general not equal to unity. In addition, as the angles may be complex, not only the ratio of the modulus but also the

[1] In figure 1 of [23] the spectra $E_k(N)$ exhibit manifestations of real EPs in the variable $N$.

phase difference can be different from the Hermitian case. The important point is, however, that equation (16) describes, up to a common factor, the only possible eigenmode at the EP.

The reduction locally of an $N$-dimensional problem to the appropriate effective two-dimensional problem around an EP is, *mutatis mutandis*, achieved along the same lines as for Hermitian operators [16]. Owing to their vanishing norm the two coalescing eigenfunctions dominate the complete set of all $N$ normalized eigenfunctions in the immediate vicinity of an EP. The expansion of the $N$-dimensional vector

$$\left|\psi_{\mathrm{EP}}^{\pm}\right\rangle = \sum \beta_k^{\pm}(\lambda)|\chi_k(\lambda)\rangle \tag{19}$$

in terms of the complete bi-orthogonal set

$$\sum |\chi_k(\lambda)\rangle\langle\chi_k(\lambda)| = 1 \qquad \lambda \neq \lambda_{\mathrm{EP}}$$

with $\beta_k^{\pm}(\lambda) = \left\langle\chi_k(\lambda)\middle|\psi_{\mathrm{EP}}^{\pm}\right\rangle$, contains virtually only two terms for $\lambda \to \lambda_{\mathrm{EP}}$. In fact, we may write

$$\lim_{\lambda\to\lambda_{\mathrm{EP}}} \begin{pmatrix} \beta_1^{\pm} \\ \vdots \\ \beta_N^{\pm} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \pm\mathrm{i}\sqrt{\frac{\cot\phi_1}{\cot\phi_2}} \\ 1 \\ 0 \\ \vdots \end{pmatrix} \tag{20}$$

up to a common factor; the two non-zero positions are given by the values $k, k+1$ for which $|\chi_k(\lambda)\rangle$ and $|\chi_{k+1}(\lambda)\rangle$ coalesce. From equations (19), (20) the effective two dimensions for any $|\psi(\lambda)\rangle$ become obvious within a small neighbourhood of $\lambda_{\mathrm{EP}}$.

We do not discuss cases where the Jordan decompositions of $h_0$ or $h_1$ or both do not yield diagonal but block matrices as this does not affect the local behaviour at an EP. This should not be confused with the fact that in all cases an EP of the full problem $H_0 + \lambda H_1$ (or $h_0 + \lambda h_1$) is characterized by a non-diagonal matrix $J$ (see equation (14)) of its Jordan decomposition.

## 4. Examples

While there are various physical reasons to consider non-Hermitian operators, we here focus on the simple mechanical model introduced in section 2. Note that the model can be easily translated into a corresponding electronic setting using two coupled R–L–C circuits. EPs can always be found for some complex values of the pair $(\omega, f)$, but a complex value of the spring constant $f$ does not appear physical. This is in contrast to quantum mechanical cases discussed previously [4] where dissipation is often described by an effective complex interaction. In the classical model we therefore introduce the damping term of the coupling denoting its strength by the real constant $g$. For given values of $\omega_j$ and $k_j$ we determine $g$ such that an EP occurs at a real value of $f$. The associated two coalescing energies are then complex describing a damped oscillation being sustained by the driving force.

The particular model reduces the resultant of equations (4) and (5) to a polynomial of fifth order in $f$ which is readily solved. The symmetry of the model implies that an EP at the pair $(\omega_{\mathrm{EP}}, f_{\mathrm{EP}})$ is always associated with an EP at $(-\omega_{\mathrm{EP}}^*, -f_{\mathrm{EP}})$; we focus our attention on positive $f$, i.e. a repulsive spring, and the physical requirement $\mathrm{Im}\,\omega_{\mathrm{EP}} < 0$ implying proper damping. Obviously, EPs can occur only if either $\omega_1 \neq \omega_2$ or $k_1 \neq k_2$ or both as otherwise a genuine degeneracy is found for $f = g = 0$.

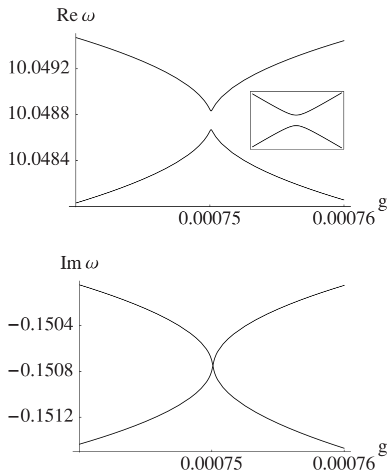


**Figure 2.** Real and imaginary parts of two repelling levels as a function of the coupling damping $g$. The parameters are $\omega_1 = \omega_2 = 10, k_1 = 0.2, k_2 = 0.1$. The spring constant $f = 1.005$ is chosen such that an EP occurs at $g = 0.000\,75$ close to the real axis. The units are arbitrary. The inset illustrates schematically a typical level repulsion for a symmetric matrix.

To get a good understanding for the EPs we first turn our attention to the behaviour of the eigenvalues of $\mathrm{i}\mathcal{M}$ in equation (2) as functions of $f$ and $g$; the eigenvalues are the solutions of equation (4). In figure 2 the real and imaginary parts of two levels coalescing at an EP very close to the real $g$-axis are plotted versus $g$ using for $f$ a real fixed value chosen such that an EP occurs in the vicinity of $g = 0.000\,75$.

The level repulsion of the real parts and the expected crossing [4] of the imaginary parts are distinctly different in shape from the usual appearance for Hermitian matrices. The cusp originates from the plain square root behaviour of the singularity, i.e. the difference between the two levels is controlled by $\sim\sqrt{\lambda - \lambda_{\mathrm{EP}}}$ (in figure 2 $\lambda \equiv g$); this is in contrast to the two complex conjugate EPs occurring in the Hermitian case where this difference is controlled by $\sim\sqrt{(\lambda - \lambda_{\mathrm{EP}})(\lambda - \lambda^*_{\mathrm{EP}})} = \sqrt{(\lambda - \mathrm{Re}\,\lambda_{\mathrm{EP}})^2 + (\mathrm{Im}\,\lambda_{\mathrm{EP}})^2}$ and hence produces a smooth approach. For illustration a typical shape of the latter is drawn schematically in the inset. The deviation from a Hermitian case is even more dramatic when the two levels are plotted against $f$ for a fixed $g = 0.000\,75$ as illustrated in figure 3. Yet, the pattern is understood by the same mechanism being a square root branch cut running along the real $f$-axis and having a branch point at $f \approx 1$.

As we deal with a classical system we now turn to the behaviour of the complex amplitudes $q_1(\omega)$ and $q_2(\omega)$ of equation (3). The overall oscillatory time behaviour is of no interest, we rather concentrate on the modulus and the phase difference. In general these complex amplitudes depend on the amplitudes $c_j$ of the driving force by equation (3). However, as discussed in the previous section, at the EP there is only one mode possible given by equation (16) up to a global constant factor. In other words, at the EP the ratio of the amplitudes of the two coalescing modes is given by $\mathrm{i}\sqrt{\cot\phi_1/\cot\phi_2}$ which is a function of only $\mathcal{M}_0$ and $\mathcal{M}_1$ and is independent of a driving force, i.e. of the $c_j$. At close distance to $\omega_{\mathrm{EP}}$ the correct value for the ratio must therefore approximately be attained.

This is well demonstrated in figure 4 where the moduli and phases are plotted against the driving frequency for the same parameters as in the previous figures giving rise to an EP at $\omega_{\mathrm{EP}} = 10.05 - 0.15\mathrm{i}$. The top drawings are the moduli of the amplitudes with the left one referring to $c_1 = \mathrm{i}, c_2 = 1$ and the right one to $c_1 = -\mathrm{i}, c_2 = 1$; below are the respective

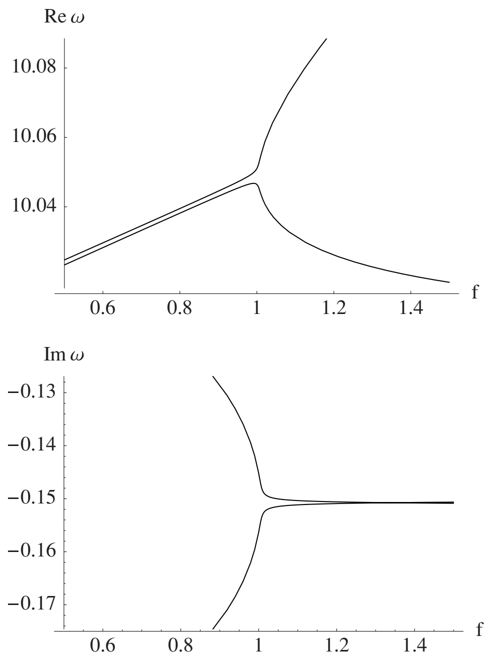


**Figure 3.** Real and imaginary parts of two repelling levels as a function of the spring constant $f$ for fixed value $g = 0.000\,75$. The parameters are the same as in figure 2.

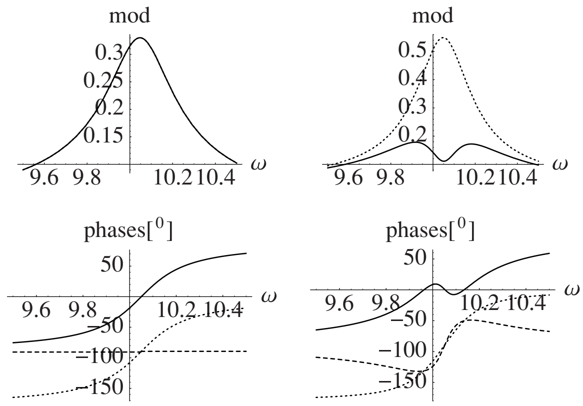


**Figure 4.** Top: modulus of the amplitudes $q_1(\omega)$ (solid line) and $q_2(\omega)$ (dotted line) for different driving amplitudes (see text); $|q_1(\omega)|$ agrees perfectly with $|q_2(\omega)|$ on the top left. Bottom: the respective phases indicated in degrees. The dashed line is the phase difference between the amplitudes, for convenience the negative value is plotted. The parameters are the same as in figure 2.

phases. The former choice (left column in figure 4) is driving the two masses with equal strength but with a leading phase of $\pi/2$ for the first mass. From the drawing we see that $q_1(\omega)/q_2(\omega) \approx +\mathrm{i}$ through the whole resonance. This value is almost equal to the exact value being $q_1(\omega_{\mathrm{EP}})/q_2(\omega_{\mathrm{EP}}) = 0.0049 + \mathrm{i}1.000\ldots$ (see (21) below). If, however, the 'incorrect' input is enforced like the lagging phase (right column), there is more variation in the response.

Yet, at the resonance the phase difference still is $+\pi/2$, i.e. opposite to the driving force and in line with the mode at the EP, even though the ratio of the moduli is quite different from unity. If the driving force is getting closer to $\omega_{\mathrm{EP}}$, i.e. if a slightly damped excitation is used, the ratio does approach the exact value *irrespective* of the values $c_j$. In fact, after some slightly tedious but straightforward algebra the result

$$\frac{q_1(\omega_{\mathrm{EP}})}{q_2(\omega_{\mathrm{EP}})} = \frac{p_1(\omega_{\mathrm{EP}})}{p_2(\omega_{\mathrm{EP}})} = \frac{f_{\mathrm{EP}} + \omega_2^2 - 2\mathrm{i}(g + k_2)\omega_{\mathrm{EP}} - \omega_{\mathrm{EP}}^2}{f_{\mathrm{EP}} - 2\mathrm{i}g\omega_{\mathrm{EP}}}$$

$$= \frac{f_{\mathrm{EP}} - 2\mathrm{i}g\omega_{\mathrm{EP}}}{f_{\mathrm{EP}} + \omega_1^2 - 2\mathrm{i}(g + k_1)\omega_{\mathrm{EP}} - \omega_{\mathrm{EP}}^2} \tag{21}$$

is obtained yielding the numerical value indicated above for the parameters considered. While this result is obtainable analytically for the particular case of equation (6), in general one has to resort to the two-dimensional reduction by numerical means and then use equations (9), (10) and (16) to find the amplitude ratio.

## 5. Summary

The physical relevance of EPs and their observability has been discussed in the introduction. The general study of the EPs has produced new general insights: (i) the parameters for which a real spectrum switches to complex values are clearly related to the occurrence of EPs on the real axis. The instability point of the RPA in quantum mechanical many-body problems is just one case in point; (ii) in addition, the specific shape of level repulsion can be quite different from the one encountered for Hermitian matrices $H_0$ and $H_1$: instead of a smooth approach the levels approach each other in the form of a cusp. On the other hand, the topological structure, i.e. the Riemann sheet structure of the energy surfaces is independent of whether $H_0$ and/or $H_1$ are Hermitian or not; (iii) the eigenfunctions at the EP have a structure similar to the symmetric case except for the value of the ratio of the two relevant states. This changes from $\pm\mathrm{i}$ for real symmetric $H_0$ and $H_1$ to $\pm\mathrm{i}\sqrt{\cot\phi_1/\cot\phi_2}$ for the non-Hermitian case. In principle, this ratio may assume any complex value. Note also that this ratio is different for the left-hand eigenfunction at the EP.

The universal significance of the EPs is once more underlined by a simple prototype example from classical physics. While the general features of EPs for non-Hermitian $H_0$ and $H_1$ have been presented in section 3, we believe that the particular results of section 4 can be experimentally confirmed, results whose analogues have so far been implemented in sophisticated microwave cavities [17, 18], optical systems [11] and atomic spectra [9].

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